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Decision-making under uncertain conditions and fuzzy payoff matrix

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Abstract

The paper has addressed a decision-making problem under imperfect information conditions. A decision-making problem and the types of imperfect information have been defined. Fuzzy information has been distinguished among this information. The criteria of decision-making under uncertain conditions for real and fuzzy payoff matrices have been discussed. Furthermore, attention has been paid to advantages and disadvantages of fuzzy logics in the decision-making process.

Introduction

In the decision-making theory the decision-making situation is deemed to be defined by giving at least the following data:

- the set of possible decisions to be taken, where A = {a1, a2, …, a m},
- the set of possible states of nature, where Y = {y1, y2, ..., yn},
- outcomes matrix, where X = [xij]m×n, and xij denotes the outcome of making ai decision and occurrence of yj state of nature,
- the evaluation function of the outcomes' utility, where U = f(X)[1].

The situation when there are more than one possible states of nature and the probabilities of their occurrence remain unknown, and a decision maker knows all the aforementioned elements, is described as decision-making under uncertainty. In this case the decision maker may make a decision in line with one of the three main criteria quoted in literature[2]: criterion of pessimism, criterion of optimism or Laplace's criterion.

Following the criterion of pessimism a decision maker acts as if he believed that in each case the nature acts maliciously towards him/her, so whatever strategy he/she chooses, a state of nature occurs which would minimise his/her payoff. Therefore, he/she chooses such a strategy in which, from his/her point of view, in the worst state of nature the utility of his/her payoff would be the greatest. Hence, it is a strategy that provides a payoff of the greatest utility out of minimal utilities (MaxMin, in other words).

On the other hand, adopting a criterion of optimism we choose such a strategy in which it is possible to gain a maximal payoff among all the possible ones.

According to Laplace's criterion, if we do not have any knowledge with regard to occurrence's probabilities of particular states of nature, we should assume they are equal. Therefore,

[1] In case when the outcomes are given in the form of a number matrix (e.g. in monetary values), the utility function may take the form of U = X.

[2] See e.g. Miller D.W., Starr M.K.. (1971). *Practice and Theory of Decisions*. Warsaw: PWN.

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in this case we should take such a decision for which the expected value of outcomes utility is the biggest in case of equivalent probability of occurrence of all possible states of nature.

Fuzzy decision-making model

However, in decision-making practice it happens quite often that a decision-maker does not have perfect information on the states of nature or the outcomes of the decisions he/she takes. In the previous article we presented the possibilities of decision-making in case of having fuzzy information on the states of nature. This paper, in turn, focuses on the analysis of the cases when a decision-maker does not have perfect information on the outcomes matrix.[3]

We can distinguish the following types of imperfect information: inadequate information, out-of-date information, inaccurate information, fuzzy information[4].

Let us assume that we are interested in **vkt** actual value of **k** attribute that characterizes a state of a given object at **t** moment. If by **ikt** we denote the information on the value of this **k** attribute of this object of our interest at **t** moment, then we define the information as perfect if **ikt = vkt**. The information is inaccurate when **ikt = vkt** + **e** (e¹0). In case when **ikt = vkt-t** (t>0) we define the received information as out-of-date, as we obtained the information on the value of the object's attribute of our interest, but the information refers to the past period of time. On the other hand, if **ikt = vlt**, then the information is inadequate, as we obtained the information on the state of object's attribute other than the one that is of our interest (irrelevant information). In case when **ikt = Vkt ,** where **Vkt** is a fuzzy number, then the information is fuzzy.

The situation when we have inaccurate, out-of-date or inadequate information, but at the same time we know how imperfect the information is, may be described as the situation in which we have fuzzy information.

Hence, in the event we have received inaccurate information, e.g. concerning a measurement's result along with giving an acceptable error value, we can assume that the measurement's result is a fuzzy triangle number of (a-d, a, a+d) form, where: a – measurement's result, d – accuracy of measurement. Whilst in the situation when we receive the information in an estimated form on the basis of a sample of a mean value of **x̄** attribute under analysis and the determined **s** standard deviation, then we can assume that we have fuzzy information in the form of a fuzzy number represented by Gaussian membership function of the following parameters **m = x̄** and **σ = s**.

When we receive out-of-date information that concerns the value of an analysed attribute at some past point of time, we can assume that we have fuzzy information in the form of e.g. a fuzzy triangle number, whose **d** spread depends on how far this point of time is from presence, or a fuzzy number represented by Gaussian membership function and **σ** that depends on how old the information is. If, though, we have such full information on the values of the analysed attribute in the past that we are able to determine a trend, then we can assume that we have fuzzy information in the form of a fuzzy number represented by Gaussian membership function, where by **m** parameter we denote the value resulting from the trend at the point of time of our interest, and by **σ** we denote a standard forecast error.

Having inadequate information and at the same time knowing from the past the regression function that describes the dependence of the value of **k** attribute of our interest on the value of **l** attribute of which we have information, we can assume that we have fuzzy information in the form of a fuzzy number represented by Gaussian membership function and the value of **m** parameter that results from the value of regression function for the received value of **l** attribute, and **σ** attribute that is equal to the root of variance of a residual component.

Decision-making under fuzzy information conditions

In the standard decision-making case under uncertain conditions we assume that we have perfect information on a payoff matrix. In reality it is frequent, though, that we do not have such information.

Let us analyse the following case. A company ships its goods to another country by sea, and then it delivers the goods by trucks to some cities within the territory of this country. The goods are in such a small quantity that it all takes place during one round when the truck comes

[3] Przybycin Z., Forlicz S. (2013). *Decision making under conditions of fuzzy information on the states of nature, United Europe: Future prospects*. Dnipropietrovsk, pp.165-169.

[4] See e.g. Forlicz S. (1997) *Economic Aspects of Imperfect Information Gathering (outline of problems)*, [in:] Materials from 33rd Conference of Statisticians, Econometrics Specialists, Mathematicians from Southern Poland PN AE in Wrocław no 771. Wrocław, pp. 111-117.

Table 1. Payoff matrix

Weather condition / Port of unloading	Good	Changeable	Bad
A	1000	1200	2600
B	1200	1500	1800
C	1500	1600	1700

Source: Own study

by all destinations and it returns to the port. The company needs to make a decision as to the choice of one out of the three existing ports of this country as a place of delivery of their goods, taking into account the costs of one round of tour when taking the decision. The costs depend on the distance necessary to travel, as well as on weather conditions on the route.

Let us assume first that the company has accurate information on the costs of one round of tour in the form of a payoff matrix in which the amounts are given in dollars:

Applying Laplace's criterion, the best port of destination would be B port, as in this case the average cost of one round with equal probability of each weather condition is the lowest and it amounts to 1500$. On the other hand, according to criterion of optimism, the best port of destination would be A port, as in case of the best weather conditions, which an optimist hopes for, the cost of these goods' delivery from this port would be the lowest and it would amount to \$1000. Finally, applying criterion of pessimism, the company would choose C port, as in the worst case scenario of bad weather conditions, the cost of goods' delivery from this port is the lowest and it amounts to \$1700.

In reality, though, the company would not have such perfect information on the travel costs, taking into account variable fuel prices in this country, fluctuations of USD, and the fact that only three possible weather conditions were considered, so the weather conditions put into one category may slightly differ from each other, causing fluctuations in the amount of the tour's costs. Hence, we can say that under real conditions the company will have only a fuzzy matrix, which results from imperfect information.

Case 1.

The company has, determined on the basis of a random sample, estimation of the tour's costs from each port under all weather conditions, so it knows a mean value and a standard deviation for each value of a payoff matrix.

Case 2.

Due to the declining value of dollar in relation to domestic currency and a continual improvement of roads quality, the average cost of the tour, determined on the basis of historical data, might be now far from the current one. However, using historical data, the company is able to determine the functions of the trend of the tour's costs from each port and under all weather conditions and, thereby, determine the expected tour's costs and standard forecast errors.

Case 3.

The company does not have the information on the tour's costs in this country. Nevertheless, it knows the timescales needed to complete the tour from each port and under all weather conditions. Simultaneously, on the basis of numerous countries' experience, the company knows that the tour's costs are quite closely correlated with the time of the tour. It has also a regression function, estimated on the basis of many other countries, which describes the dependence of costs' amount on the time of a tour, and it knows variation of a residual component.

Let us assume now that in the analysed case we have fuzzy information - travelling costs are fuzzy triangle numbers in the form of: $(\bar{x} - s, \bar{x}, \bar{x} + s)$, where $\bar{x}$ is an average value of a tour's cost estimated on the basis of a sample, while **s** is a standard deviation of this cost (case 1).

Let us assume further the following form of a payoff matrix:

The decision-making criteria in case of a fuzzy payoff matrix are analogous to criteria used for a real payoff matrix.

The criteria of optimism and pessimism require introducing max and min relations in a fuzzy numbers set. In this study we focus only on defining these relations for a special case[5], namely:

[5] More information regarding max-min relations might be found in the publication: Łachwa A. (2001) *Fuzzy World of Sets, Numbers, Relations, Facts, Rules and Decisions* Warsaw: EXIT.

Table 1. Fuzzy payoff matrix

Port of unloading \ Weather condition	Good	Changeable	Bad
A1	(990, 1000, 1010)	(1188, 1200, 1212)	(2574,2600, 2626)
A2	(1188,1200, 1212)	(1485,1500, 1515)	(1782,1800, 1818)
A3	(1485,1500, 1515)	(1584,1600, 1616)	(1683,1700, 1717)

Source: Own study

if **A** ≤ **B,** then min (**A,B**) = **A**, and max (**A,B**) = **B.** wherein **A** ≤ **B** if, and only if, for each α, αϵ < 0,1>, the sets of α[6] level of **A** and **B** fuzzy numbers satisfy the conditions:
a(α) ≤ c(α) and b(α) ≤ d(α) where Aα = [a(α) , b(α)], Bα = [c(α) , d(α)] (fuzzy numbers were denoted by bold characters in upper case).
As fuzzy numbers of a payoff matrix satisfy the condition of comparability, so
in accordance with a criterion of optimism the best port of unloading is A1 port. The transport cost from this country would account to approx. $1000. On the other hand, the criterion of pessimism would suggest A3 port, in which case the transport cost would account to $1700.
Laplace's criterion requires defining algebraic operations for fuzzy numbers, in particular, multiplying a fuzzy number by a real number and adding fuzzy numbers.

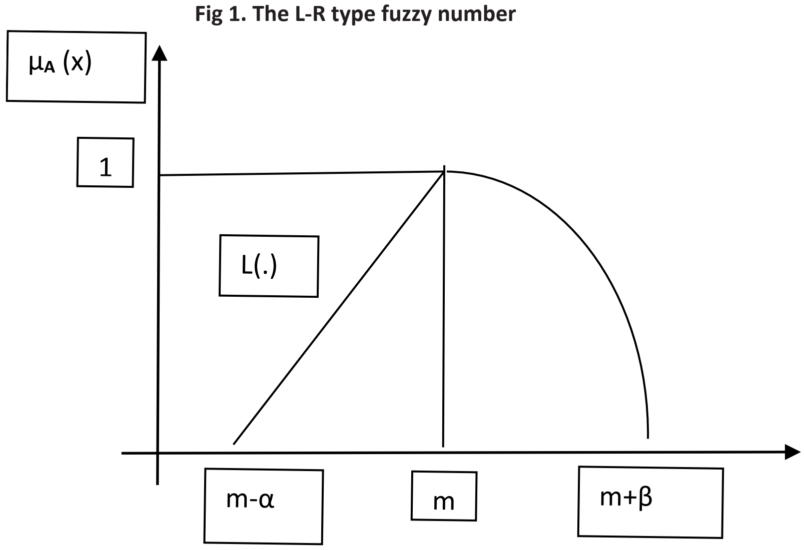
[6] The set of α level of **A** fuzzy number we define as follows: $A_\alpha = \{ x: x \epsilon R \text{ and } \mu_A(x) > \alpha \}$, where $\mu_A(x)$ is a membership function of this set.

We are going to define these operations for L-R type fuzzy numbers.
We define the membership function of **A** fuzzy number of L-R type as follows:

$$\mu\mathbf{A}(x) = \begin{cases} L(m-x/\alpha) & \text{for } x < m \\ 1 & \text{for } x = m \\ R(x-m/\beta & \text{for } x > m \end{cases}$$

where α,β > 0 then the established dispersions are in the left and right sides, wherein L(.) base function is a non-decreasing function and R(.) base function is a non-increasing function.
The L-R type fuzzy number is depicted in the drawing 1 as below.
The L-R type fuzzy number we are going to denote in the following manner: (m, α, β)LR.
The product of L-R type fuzzy number by a real number is calculated as follows:
λ (m**A**, αA, βA)LR = (λ m**A**, λ αA, λ βA)LR, λ ϵ R.
The sum of **A**, **B** R-L type fuzzy numbers we denote by a formula:
(m**A**, αA, βA)LR + (mB, αB, βB)LR = (mA + mB, αA+ αB, βA + βB)LR.

Fig 1. The L-R type fuzzy number



Source: Own study

To determine a difference between these numbers one needs only to replace the plus sign with a minus sign in the above formula.

Let us note that a sum, product of L-R-type numbers is a fuzzy number belonging to the same type. It should be clearly pointed out that the given formulae are approximate[7]. The lower α and β dispersions are, the better the approximation is.

As fuzzy triangle numbers belong also to L-R type, so with the help of the operations defined earlier we can apply Laplace's criterion for the analysed case. According to this criterion, the best port of unloading is A2 port, as the expected transport cost, on the assumption of the same probabilities of occurrence of the distinguished weather conditions for this port, is the lowest and it amounts to (1500, 15, 15), which is about 1500$.

Expressing the elements of a payoff matrix in the terms of fuzzy numbers allows considering different scenarios of the analysed decision-making problem.

Assuming that as a result of applying a selected choice criterion we receive a payoff in the form of a L-R type fuzzy number, that is $(m, \alpha, \beta)_{LR}$, then in the case of a minimised payoff, the payoff at m- α level is optimistic, whereas the payoff at m+β level is certainly pessimistic one. Overall, if a payoff is **A** fuzzy number and the membership function[8] of this number is known, then it is possible to determine the sets of α level (α intersections), which is the sets of $A\alpha = [a(\alpha) , b(\alpha)]$, and then the payoff in the amount of a(α) is optimistic, while the one in the amount of b(α) is pessimistic.

In case of payoffs' maximization, the relations are certainly inverse.

Let us note that for a L-R fuzzy number, the interval [m-α , m+β] is a zero intersection of this number (α = 0).

Having α intersections of a fuzzy number for each scenario, we determine a fuzzy risk of σ payoff according to the formula:

$\sigma = 0{,}5\,\alpha\,(b(0) - a(0))$

where [a(0), b(0)] is a zero intersection of A fuzzy number.

[7] Definitions of algebraic operations on fuzzy numbers of any types are given by e.g. Kacprzyk J. (1986). *Fuzzy Sets in the System Analysis* Warsaw: PWN.

[8] More information regarding max-min relations might be found in the publication: Łachwa A. (2001). *Fuzzy World of Sets, Numbers, Relations, Facts, Rules and Decisions* Warsaw: EXIT.

Summary

The necessary condition for taking rational decisions is the availability of perfect (precise) information in the decision-making process. It is obvious that in practice the access to this information is hindered or hardly possible. A decision maker has often imperfect information. From a practical point of view, fuzzy information is essential, as it is generated on the basis of historical data, as well as expert knowledge. As a result, the information brings also predictive advantages, which is of substantial importance in the decision-making process, as the effect of a decision taken is frequently seen only within specific time horizon.

Quantifying decision-making process in the fuzzy terms allows analysing different scenarios, thus facilitating flexible decision-making. It is also quite important that fuzzy information creates the opportunity to measure a fuzzy risk of a decision taken.

It should be clearly underlined that the results received with the use of fuzzy logics are of subjective nature. It is due to expert knowledge involved in the identification process.

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